A Banach Algebra Approach to Noncommutative Integration

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Abstract

We review the basic concepts of stochastic integration and reformulate it in terms of a Banach four-normed *-algebra with the associative product given by the stochastic covariation. We show that this nonunital algebra has two nilpotent first and second order *-ideals with the C*-normed quotient algebra, being a generalization of the C*-algebra corresponding to the only nontrivial operator norm. The noncommutative generalization of this algebra called B*-algebra leads to the *-algebraic theory of quantum stochastic integration developed in [1-4]. The main notions and results of classical and quantum stochastic analysis are reformulated in this unifying approach. The general Le´vy process is defined in terms of the modular B*-Ito algebra and the corresponding quantum stochastic master equation on the predual space of a W*-algebra is derived as a noncommutative version of the Zakai equation driven by the process. This is done by a noncommutative analog of the Girsanov transformation, which we introduce in full generality here.

Quantum adapted integrable processes

Let $\left(\mathfrak{d}^t\right)$ be an increasing family of unital pre-C*-algebras $\mathfrak{d}^s \subseteq \mathfrak{d}^t\ \forall 0 \leq s \leq t$ embedded into their L^1-completions $\mathfrak{l}^s \subseteq \mathfrak{l}^t$ w.r.t. a faithful state $\mathbf{1} \in \mathfrak{l}^t$ defining the pairing

$$\langle \mathbf{d}\mathbf{q}\mathbf{d}^* | \mathbf{p} \rangle = \langle \mathbf{d}\mathbf{q} | \mathbf{p}\mathbf{d} \rangle = \left\langle \mathbf{q} | \mathbf{d}^\dagger \mathbf{p}\mathbf{d} \right\rangle \quad \forall \mathbf{d}, \mathbf{q} \in \mathfrak{d}^t, \mathbf{p} \in \mathfrak{l}^t$$

with real values $\langle \mathbf{q}\mathbf{q}^* | \mathbf{p} \rangle \geq 0$ on the positive $\mathbf{p} = \mathbf{d}^\dagger \mathbf{d}$ dominated by $\mathbf{1} = \mathbf{1}^\dagger \in \mathfrak{l}$ for a faithful $\langle \mathbf{q}\mathbf{q}^* | \mathbf{1} \rangle$. We assume that $\mathbf{1} \in \mathfrak{l}$ admits the conditional expectation $\mathsf{E}^t : \mathfrak{d} \mapsto \mathfrak{d}^t$ on $\mathfrak{d} = \cup \mathfrak{d}^t$, $\mathsf{E}^r \circ \mathsf{E}^t = \mathsf{E}^r\ \forall\ r, t \in \mathbb{R}_+$ s.t.

$$\mathsf{E}^t(\mathbf{1}) = \mathbf{1},\ \mathsf{E}^t\left(\mathbf{d}^\dagger \mathbf{p}\mathbf{d}\right) = \mathbf{d}^\dagger \mathsf{E}^t(\mathbf{p})\,\mathbf{d} \quad \forall \mathbf{p} \in \mathfrak{l}, \mathbf{d} \in \mathfrak{l}^t_\natural.$$

Let $\left(\mathcal{D}^t\right)$ be projective family of continuous *adapted* functions $q(s) \in \mathfrak{d}^s$ on $[0, t[$ and $\mathcal{D} = \curlyvee_{t>0} \mathcal{D}^t$. An adapted process $\mathsf{X} : t \mapsto \mathsf{X}_t \in \mathfrak{l}^t$ is called *locally integrable* if $\mathsf{X} \in \mathcal{L}$, where $\mathcal{L} = \mathcal{D}^\sharp$ is the *dominated* (by $\mathbf{1}(t) = \mathbf{1}$) dual space of locally L^1-adapted functions $p(t) \in l^t$ w.r.t. the integral pairing

$$\langle q | p \rangle = \int \langle q(t) | p(t) \rangle \,\mathsf{d}t \ \ \forall q \in \mathcal{D}, p \in \mathcal{L}.$$

The quantum Itô semimatingales

We consider adapted *quantum Itô processes* $\mathsf{X} = (\mathsf{X}_t)$ formally defind as the *special semimartingales*

$$\mathsf{X}_t - \mathsf{X}_r = \int_r^t \boldsymbol{k}(s) \centerdot \mathrm{d}\boldsymbol{B}(s) \equiv \Lambda_r^t(\boldsymbol{k}).$$

Here $\boldsymbol{k}(t) = \{\mathcal{X} \ni \varkappa \mapsto k(t,\varkappa)\}$ are adapted integrands indexed by a measurable set $\mathcal{X}$ with an isolated point $\oslash \in \mathcal{X}$ invariant under a reflection $\varkappa \mapsto -\varkappa$ $\forall \varkappa = -(-\varkappa) \in \mathcal{X}$ and a l.c.s. $\mathcal{X}_\circ = \mathcal{X} \setminus \oslash$ s.t.

$$\Lambda_r^t(\boldsymbol{k}) = \int_r^t \int_{\mathcal{X}_\circ} k(s,\varkappa)\,\mathrm{dB}(t,\mathrm{d}\varkappa) + \int_r^t k(s,\oslash)\,\mathrm{d}s,$$

where $\mathsf{B}(t,\cdot)$ is a martingale-valued measure on $\mathcal{X}_\circ$ and

$$\mathsf{E}^r\left[\Lambda_r^t(\boldsymbol{k})\right] = \int_r^t k(s,\oslash)\,\mathrm{d}s \equiv \epsilon[\mathsf{X}_t] - \epsilon[\mathsf{X}_r]$$

is given by $\mathsf{B}(t,\oslash) = t\mathsf{1} \equiv B_-^+(t)$ as a.c. variation of

$$\epsilon[\mathsf{X}]_t = \mathsf{X}_0 + \int_0^t k_+^-(s)\,\mathrm{d}s$$

for $k_+^-(t) := k(t,\oslash) \equiv \epsilon[\boldsymbol{k}(t)]$. We assume that

$$\mathsf{B}^\star(t,\mathrm{d}\varkappa) := \mathsf{B}(t,-\mathrm{d}\varkappa)^\dagger = \mathsf{B}(t,\mathrm{d}\varkappa)\ \ \forall \mathrm{d}\varkappa \in \mathfrak{F}(\mathcal{X}).$$

The quantum stochastic covariation

Assuming instead of independence only commutativity

$$\mathsf{X}_t \mathrm{d}\mathsf{B}(t,\cdot) = \mathrm{d}\mathsf{B}(t,\cdot)\,\mathsf{X}_t \quad \forall \mathsf{X}_t \in \mathfrak{d}^t,$$

we have $\Lambda(\boldsymbol{k})^\dagger = \Lambda(\boldsymbol{k}^\star)$, where $\boldsymbol{k}^\star(t,\varkappa) = \boldsymbol{k}(t,\varkappa)^\dagger$, and threfore $\mathsf{X}_t^\dagger = \mathsf{X}_0^\dagger + \Lambda_0^t(\boldsymbol{k}^\star)$.

Moreover, we shall assume that the *stochastic covariation*, defined if $\mathsf{X}_t, \mathsf{Y}_t \in \mathfrak{d}^t$ for all $t \in \mathbb{R}_+$ by

$$[\mathsf{X};\mathsf{Y}]_t := \int_0^t \left[\mathrm{d}(\mathsf{X}_s \mathsf{Y}_s) - (\mathrm{d}\mathsf{X}_s)\,\mathsf{Y}_s - \mathsf{X}_s(\mathrm{d}\mathsf{Y}_s)\right],$$

can be written in terms of an *associative* Itô product

$$(\boldsymbol{k}\,\centerdot\,\mathrm{d}\boldsymbol{B})(\boldsymbol{h}\,\centerdot\,\mathrm{d}\boldsymbol{B}) = (\boldsymbol{k}\cdot\boldsymbol{h})\,\centerdot\,\mathrm{d}\boldsymbol{B}$$

of noncommuting $\mathrm{d}\mathsf{X} = \boldsymbol{k}\,\centerdot\,\mathrm{d}\boldsymbol{B}$ and $\mathrm{d}\mathsf{Y} = \boldsymbol{h}\,\centerdot\,\mathrm{d}\boldsymbol{B}$ as

$$[\mathsf{X};\mathsf{Y}]_t = \int_0^t \mathrm{d}\mathsf{X}\mathrm{d}\mathsf{Y} = \int_0^t (\boldsymbol{k}\cdot\boldsymbol{h})\,\centerdot\,\mathrm{d}\boldsymbol{B}.$$

In other words, the quantum Itô semimartingales form a nonunital †-algebra w.r.t. $[\cdot;\cdot]$ given by an associative quantum Itô $\star$-algebra of the corresponding QS integrands $\boldsymbol{k}(t)$ as the QS derivatives $\boldsymbol{D}_{\mathsf{X}}(t)$ of X at t.

The generalized H-Schmidt module

A right $\mathfrak{d}$-module $\mathfrak{h}$ is called Hilbert-Schmidt (HS) if it is Hilbert space with respect to the scalar product

$$\langle \mathsf{kq}|\mathsf{h}\rangle := \left\langle \mathsf{q}|\mathsf{k}^{\dagger}\mathsf{h}\right\rangle \quad \forall \mathsf{k},\mathsf{h}\in\mathfrak{h},\mathsf{q}\in\mathfrak{d}.$$

given by the left acion $h^{\circ}:\mathsf{q}\mapsto\mathsf{hq}$ of $\mathsf{h}=\mathsf{h}^{\circ}1\equiv\mathsf{h}^{\circ}$ on $\mathfrak{d}$ with the adjoint action of $\mathsf{h}^{\dagger}\equiv\mathsf{h}_{\circ}$ into $\mathfrak{d}^{\natural}$ defining the $\mathfrak{l}$-valued inner product $\mathsf{h}^{\dagger}\mathsf{k}:=\mathsf{h}_{\circ}\mathsf{k}^{\circ}$.For the nonunital $\mathcal{D}=\mathcal{L}_{\natural}$ the right HS module K is defined in the generalized sense as the space of left adjointable operators $k_{+}:\mathcal{L}_{\natural}\to\mathsf{K}_{\natural}$, into the Frechet space $\mathsf{K}_{\natural}=\mathsf{K}\mathcal{D}$ dense w.r.t. $\|k^{\circ}d\|^{2}=\langle k^{\circ}d|k^{\circ}d\rangle$ in the Hilbert space $\mathsf{H}\subseteq\mathsf{K}$. Thus, $\mathcal{K}^{\circ}\equiv\mathsf{K}$ is right and $\mathcal{K}_{\circ}\equiv\mathsf{K}^{\dagger}$ is left $\mathcal{D}$-module with adjoint inner products

$$(k^{\circ}|h^{\circ}):=k_{\circ}h^{\circ}\equiv(h_{\circ}|k_{\circ})^{\dagger}\in\mathcal{D}^{\natural}\;\forall k^{\circ}=k_{\circ}^{\dagger},h^{\circ}=h_{\circ}^{\dagger}.$$

Note that since $c^{\dagger}=c^{*}$ for any central $c\in\mathcal{C}(\mathcal{D})$, $c_{\circ}^{\circ}\mathsf{k}=\mathsf{k}c\;\forall\mathsf{k}\in\mathsf{K}$ and $c(t)\in\mathfrak{c}^{t}$ is naturally amalgmated into $\mathfrak{L}\left(\mathfrak{h}^{t}\right)$ for $\mathfrak{h}^{t}=\mathsf{K}(t)$. In particular, $\mathsf{K}_{\natural}=\Upsilon_{t>0}\mathsf{H}^{t}$ for $\mathsf{H}^{t}=\mathsf{H}1^{t}=\mathsf{K}1^{t}=\mathsf{K}^{t}$, where $1^{t}(s)=1$ for $s<t$ and $1^{t}(s)=0$ otherwise, and both K and $\mathfrak{L}(\mathsf{K})$ are represented by locally L^{2} and L^{∞} adapted functions $k^{\circ}(t)\in\mathfrak{h}^{t}$ and $a_{\circ}^{\circ}(t)\in\mathfrak{L}\left(\mathfrak{h}^{t}\right)$.

The Itô $\star$-algebra of an HS bi-module

Given a $\dagger$-subalgebra $\mathcal{M} \subseteq \mathfrak{L}(\mathsf{K})$ of adjointable opertors on $\mathsf{K} = \mathcal{K}_{+}$, we extend it to a nonunital $\star$-algebra $\mathcal{A} = \mathcal{L}\times\mathcal{K}_{\circ}\times\mathcal{K}^{\circ}\times\mathcal{M}$ of the quadruples $\boldsymbol{a} = \left(a_{+}^{-}, a_{\circ}^{-}, a_{+}^{\circ}, a_{\circ}^{\circ}\right)$ with $\boldsymbol{a}^{\star} = \left(a_{+}^{-\dagger}, a_{+}^{\circ\dagger}, a_{\circ}^{-\dagger}, a_{\circ}^{\circ\dagger}\right)$ and Itô product

$$\boldsymbol{a}\cdot\boldsymbol{b} = \left(a_{\circ}^{-}b_{+}^{\circ},\ a_{\circ}^{-}b_{\circ}^{\circ},\ a_{\circ}^{\circ}b_{+},\ a_{\circ}^{\circ}b_{\circ}^{\circ}\right).$$

It is induced by the matrix representation $\boldsymbol{a}\cdot\boldsymbol{b} \mapsto \mathbf{ab}$ in the $\ddagger$-algebra $\mathfrak{L}(\mathbb{K})$ of the adjointable operators

$$\mathbf{a} = \begin{bmatrix} 0 & a_{\circ}^{-} & a_{+}^{-} \\ 0 & a_{\circ}^{\circ} & a_{+}^{\circ} \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{a}^{\ddagger} = \boldsymbol{I}\mathbf{a}^{\dagger}\boldsymbol{I},\ \boldsymbol{I} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

on the pseudo-HS module $\mathbb{K} = \mathcal{L}_{\natural} \oplus \mathsf{K} \oplus \mathcal{L}$ of $\mathbf{k} = (q, k_{\circ}, p)^{\ddagger}$. with respect to the $\mathfrak{l}$-valued inner product

$$\mathbf{k}^{\ddagger}\mathbf{k} := qp^{\dagger} + k_{\circ}k_{\circ}^{\dagger} + pq^{\dagger} \equiv \mathbf{k}^{\dagger}\boldsymbol{I}\mathbf{k}.$$

The $\star$-algebra $\mathfrak{A} \sim \mathcal{A}$ with $\mathcal{M} = \mathfrak{L}(\mathsf{K})$ will be called the Itô agebra of the right module $\mathfrak{k}$, denoted $\mathfrak{l}(\mathsf{K})$. Note that the operators $\mathbf{a}$ are continuous on each $\mathbb{K}^{t} = \mathbb{K}\mathbf{1}^{t}$ w.r.t.

$$\|\mathbf{k}\|^{+} = \|q\|, \|\mathbf{k}\|^{\circ} = \|k^{\circ}\|, \|\mathbf{k}\|^{-} = \|p\|.$$

The completeness of Itô algebras

Let us fix $\mathsf{K}^t_\natural = \mathsf{H} = \mathsf{K}^t$ as Hilbert $\mathcal{D}$-module with $\|h^\circ\| = \sqrt{\langle 1|h_\circ h^\circ\rangle} = \|h_\circ\|$. Then the algebra $\mathfrak{l}(\mathsf{H})$ is complete w.r.t. the uniform topology induced by a quadruple $(\|\cdot\|^\mu_\nu)^{\mu=-,\circ}_{\nu=+,\circ}$ of the seminorms $\|\boldsymbol{a}\|^\circ_\circ = \|a^\circ_\circ\|$,

$$\|\boldsymbol{a}\|^\circ_+ = \left\|a^\circ_+\right\|, \ \|\boldsymbol{a}\|^-_\circ = \left\|a^-_\circ\right\|, \ \|\boldsymbol{a}\|^-_+ = \left\|a^-_+\right\|.$$

Note that $(\|\boldsymbol{a}\|^\mu_\nu) = 0 \Leftrightarrow \boldsymbol{a} = 0$ and the *B*-property*

$$\|\boldsymbol{a}\cdot\boldsymbol{a}^\star\|^-_+ = \|\boldsymbol{a}\|^-_\circ \|\boldsymbol{a}^\star\|^\circ_+ , \quad \|\boldsymbol{a}\cdot\boldsymbol{a}^\star\|^\circ_\circ = \|\boldsymbol{a}\|^\circ_\circ \|\boldsymbol{a}^\star\|^\circ_\circ .$$

A $\star$-subalgebra $\mathfrak{B}$ of $\mathfrak{l}(\mathsf{H})$ is called *B*-algebra* if it is complete in $\|\cdot\|^\bullet_\bullet$ and the *Itô B*-algebra* if it is an extension $\mathfrak{B} \supseteq \mathcal{L}\mathbf{e}^+_-$ of its projection $\mathcal{L} = \mathfrak{B}^-_+$ given by a unital modular subalgebra $\mathcal{G} = \mathcal{L}_\natural \subseteq \mathcal{D}$ represented also in $\mathfrak{B}$ by $g^\circ_\circ \in \mathfrak{B}^\circ_\circ \subseteq \mathfrak{L}(\mathsf{H})$. Here $\mathbf{e}^+_-$ stands for the nilpotent matrix representing $\boldsymbol{e}^+_- = (1, 0, 0, 0)$.

The general and abstract B*-algebras

The *abstract B*-algebra* $\mathcal{A}$ is defined similarly to C*-algebra as a Banach $\star$-algebra with respect to the quadruple seminorm separating $\mathcal{A}$ in the sense $\|\boldsymbol{a}\|_{\nu=+,\circ}^{\mu=-,\circ} = 0 \Rightarrow \boldsymbol{a} = 0$ and satisfying the four inequalities

$$(\|\boldsymbol{a}\cdot\boldsymbol{b}\|_{\nu}^{\mu} \leq \|\boldsymbol{a}\|_{\circ}^{\mu}\|\boldsymbol{b}\|_{\nu}^{\circ})_{\nu=+,\circ}^{\mu=-,\circ}$$

for all $\boldsymbol{a},\boldsymbol{b}\in\mathcal{A}$ with the $\star$-property $\|\boldsymbol{a}^{\star}\|_{-\nu}^{\mu} = \|\boldsymbol{a}\|_{-\mu}^{\nu}$ and the two equalities of the B*-property for $\boldsymbol{a}=\boldsymbol{b}$. The *abstract Itô algebra* $\mathcal{A}$ *with* $\mathfrak{l}=\mathfrak{d}^{\natural}$ naturally has B*-norm if $1\in\mathfrak{d}$, defined by: (i) $\exists$ an embedding $E(\mathfrak{l})\subseteq\mathcal{A}$ of the projection $\mathfrak{l}=\epsilon(\mathcal{A})$ into $\mathcal{A}$ as $\epsilon\circ E=\mathbf{id}$ s.t.

$$E\left(l^{\dagger}\right)=E(l)^{\star},\quad E(l)\mathcal{A}=0=\mathcal{A}E(l)\quad\forall l\in\mathfrak{l},$$

(ii) The triviality of the $\star$-ideal $\mathfrak{I}=\{\boldsymbol{b}\in\mathcal{A}\}$ s.t. $\forall l\in\mathfrak{l}$

$$l(\boldsymbol{b})=l(\boldsymbol{a}\cdot\boldsymbol{b})=l(\boldsymbol{b}\cdot\boldsymbol{c})=l(\boldsymbol{a}\cdot\boldsymbol{b}\cdot\boldsymbol{c})=0\;\forall\boldsymbol{a},\boldsymbol{c}.$$

Theorem There exists unique, up to $\mathbf{U}^{\ddagger}=\mathbf{U}^{-1}$, isometric $\star$-representation $\mathbf{i}=\left(i_{\nu}^{\mu}\right)_{\nu=+,\circ}^{\mu=-.\circ}$ of $\mathcal{A}$ in the operator algebra $\mathfrak{l}(\mathfrak{k})$ of a minimal HS $\mathfrak{d}$-module $\mathfrak{k}$ generated by $i_{+}^{\circ}(\mathcal{A})=i_{\circ}^{-}(\mathcal{A})^{\dagger}$ with $i_{\circ}^{\circ}(\mathcal{A})\subseteq\mathfrak{L}(\mathfrak{k})$ and $i_{+}^{-}=\epsilon$ and $\mathbf{U}=\mathbf{I}+\mathbf{K}$ given by $\mathbf{K}\in\mathfrak{l}(\mathfrak{k}\rightarrow\mathfrak{k}')$ for the $\mathfrak{k}'\simeq\mathfrak{k}$.

The germ-algebra and its commutant

Thehe *germ-algebra* $\mathfrak{G}=\mathcal{G}\mathbf{I}+\mathfrak{B}$ *over* $\mathcal{G}$ *for the Itô* $\star$-*algebra* $\mathfrak{B}$ is well-defined by the triangular ‡-representation $\mathbf{G}=g\mathbf{I}+\mathbf{b}$ for $g\in\mathcal{G}$ and $\mathbf{b}\in\mathfrak{B}$ in terms of $(g,\boldsymbol{b})$ with $\boldsymbol{b}=\left(B,b_{+}^{\circ},b_{\circ}^{-},b_{+}^{-}\right)$, where $B=b_{\circ}^{\circ}+g_{\circ}^{\circ}$, by

$$(g,\boldsymbol{b})\cdot\left(g^{\dagger},\boldsymbol{b}^{\star}\right)=\left(gg^{\dagger},\boldsymbol{b}\dotplus\boldsymbol{b}^{\star}\right),$$

$$\boldsymbol{b}\dotplus\boldsymbol{b}^{\star}=\boldsymbol{b}\cdot\boldsymbol{b}^{\star}+\left(0,b_{+}^{\circ}g^{\dagger},gb_{+}^{\circ\dagger},b_{+}^{-}g^{\dagger}+gb_{+}^{-\dagger}\right).$$

Let $\mathfrak{B}\subseteq\mathfrak{l}(\mathfrak{k})$ be an operator Itô algebra representing on $\mathbb{k}$ the general Itô algebra $\mathcal{B}$ with $\mathfrak{B}_{+}^{-}\subseteq\mathfrak{d}^{\natural}$. It is called the *achieved* Itô algebra on $\mathbb{k}$ if $\mathfrak{G}=\mathcal{G}\mathbf{I}+\mathfrak{B}$ is the *adjointable commutant* of another gerrm-algebra $\mathfrak{F}=\mathcal{F}\mathbf{I}+\mathfrak{A}$ over a multiplier †-subalgebra $\mathcal{F}=\mathfrak{L}\left(\mathfrak{A}_{+}^{-}\right)$:

$$\mathbf{b}\in\mathfrak{B}\Leftrightarrow[\mathcal{G}\mathbf{I}+\mathbf{b},\mathcal{F}\mathbf{I}+\mathfrak{A}]=0.$$

Note that the commutant of $\mathfrak{F}=\mathcal{F}\mathbf{I}+\mathfrak{A}$ with $\mathcal{F}=\mathfrak{L}(\mathfrak{d})$ is the germ over $\mathcal{G}=\mathfrak{L}(\mathcal{C})$ with $\mathfrak{B}_{+}^{-}=\mathcal{C}^{\natural}$. In particular, $\mathfrak{A}=\mathfrak{l}(\mathfrak{k})$ is achieved and its germ-commutant is given by the trivial achieved Itô algebra $\mathfrak{B}=\mathcal{L}\mathbf{e}_{-}^{+}$ with $\mathcal{L}=C^{\natural}$ embedded into $\mathfrak{L}(\mathbb{K})$ by the nilpotent matrix $\mathbf{e}_{-}^{+}$ representing the death element $\boldsymbol{e}_{-}^{+}=(1,0,0,0)$ projected onto $1\in\mathfrak{l}$ by ϵ.

The Lévy-Itô algebra of thermal noise

If $\mathbb{K}$ is genrated by the germ-algebra $\mathfrak{F} = \mathcal{F}\mathbf{I} + \mathfrak{A}$ on all $\mathbf{c} = (c, 0, 0)^{\ddagger}$ for $c \in \mathfrak{c}(\mathfrak{d})$, then the germ-commutant $\mathfrak{G}$ is faithfully given on the right $\mathcal{C}\mathfrak{G} \equiv \boldsymbol{K}_{\flat}$ of $\boldsymbol{c} = \mathbf{c}^{\ddagger}$ by a $\flat$-algebra $\boldsymbol{K}_{\flat} = \mathcal{C} \times \mathcal{K}_{\flat} \times \mathcal{C}^{\natural}$ embedded into $\mathbb{k}^{\ddagger} = \mathfrak{d} \times \mathcal{K} \times \mathfrak{d}^{\natural}$. The product $\boldsymbol{k} \cdot \boldsymbol{k}^{\flat}$ and $(c, k, l)^{\flat} = \left(c^{*}, k^{\flat}, l^{*}\right)$, where $c \in \mathcal{C}$, $k \in \mathcal{K}_{\flat}$, $l \in \mathcal{C}^{\natural}$, are defied $\forall \boldsymbol{k} = \boldsymbol{c}\mathbf{G}$ by

$$\boldsymbol{k}^{\flat} = (c^{*}, 0, 0)\, \mathbf{G}^{\ddagger}, \quad \boldsymbol{k} \cdot \boldsymbol{k}^{\flat} = (cc^{*}, 0, 0)\, \mathbf{G}\mathbf{G}^{\ddagger} \;\; \forall \mathbf{G} \in \mathfrak{G},$$

$$(\boldsymbol{k} \cdot \boldsymbol{h}^{\flat} | \boldsymbol{k}) := \boldsymbol{k}\, (\boldsymbol{k} \cdot \boldsymbol{h})^{\dagger} \equiv (\boldsymbol{k} | \boldsymbol{k} \cdot \boldsymbol{h}) \in \mathcal{C}^{\natural} \;\; \forall \boldsymbol{h}, \boldsymbol{k} \in \boldsymbol{K}_{\flat}\, .$$

The corresponding Itô $\ddagger$-algebra $\mathfrak{B} = \mathfrak{G} \ominus \mathcal{G}$, given by the pairs $b = (k, l)$ of $\mathcal{K}_{\flat} \times \mathcal{C}^{\natural} \equiv \mathfrak{b}^{\ddagger}$ embedded into $\boldsymbol{K}_{\flat}$ as $(0, \mathfrak{b})$, is called the *thermal noise Lévy-Itô algebra*.

Note that $\boldsymbol{K}_{\flat}(\varkappa) = \mathbb{C} \times \mathcal{K}_{\flat}(\varkappa) \times \mathbb{C}$ is right Krein algebra given on the spectrum $\mathcal{X}$ of $\mathcal{C}$ by a right Hilbert (Tomita) algebra $\mathcal{K}_{\flat}(\varkappa)$. However, unlike Tomita, we do not assume that the subalgebra $\mathcal{K}_{\flat}^{2}$ is dense in $\mathcal{K}_{\flat}$ for any $\varkappa \in \mathcal{X}$. In particular, $k \cdot h = 0 \; \forall \; k, h$ in the *Heisenberg modular algebra* $\mathfrak{B}$ describing a quantum Wiener noise by $\mathcal{K}_{\flat}$. The Tomita case ($\mathcal{K}_{\flat} \ni 1$, say) corresponds to a quantum Poisson noise (with finite $\lambda = \langle 1 | l \rangle$).

The Itô algebra of adapted integrands

Take $\mathcal{C} = \cup C^t(\mathcal{X})$ as the projective limit $\mathcal{C} = \curlyvee_{t>0}\mathcal{C}^t$ of the increasing unital quotients $\mathcal{C}^t := C\left(\mathcal{X}^t\right) \prec C_0(\mathcal{X})$ on the compacts $\mathcal{X}^t = \{\varkappa \in \mathcal{X} : \tau(\varkappa) \leq t\}$ of $\mathcal{X} = \cup\mathcal{X}^t$ by a conti#nuous surjection $\tau : \mathcal{X} \to \mathbb{R}_+$ w.r.t. a nonatomic measure $\langle 1|c\rangle = \int c(\varkappa)\,\mathrm{d}\varkappa$. It defines the dominating identity $1 \in \mathfrak{l}$ and $\mathcal{C}^\natural = \curlyvee L^1\left(\mathcal{X}^t\right)$. Assume that $\mathcal{K}_\flat^t = \curlyvee\mathcal{K}_\flat^t$ given by the unital $\flat$-subalgebras $\mathcal{K}_\flat^t \subseteq C\left(\mathcal{X}^t \to K_\flat^t\right)$ of L^2-functions $k(\varkappa) = \kappa_\varkappa \in K_\flat^{\tau(\varkappa)}$, $k^\flat(\varkappa) = \kappa_\varkappa^\flat$ into an increasing family $\left(K_\flat^t\right)$ of unital right Hilbert algebras $K_\flat^t \subseteq H^t$ in the sections $\mathcal{K}(\varkappa) = H^{\tau(\varkappa)}$ of increasing Hilbert spaces $\mathcal{K}^t \subseteq L^2\left(\mathcal{X}^t \to K^t\right)$ for $\mathcal{K} = \curlyvee\mathcal{K}^t$. This deines the thermal Itô algebra $\mathfrak{B}$ *of adapted integrands* $\mathbf{K}(t) \in \left(\mathbb{K} \times K_\flat^\dagger \otimes K_\flat \times L\right)^t$ with $L^t = L^1\left(X^t\right)$ and increasing vN algebras $\mathbb{K}^t$ generated on H^t by operators $\mathsf{K} : h \mapsto h \cdot \kappa \equiv h\mathsf{K}$ for all $\kappa \in \mathsf{K}_\flat$, $h^t \in H^t$ if the sections $\mathcal{X}(t) = \tau^{-1}(t) \equiv X^t$ are projectively increasing, $X^s \preccurlyeq X^t\ \forall s \leq t$, s.t. $L^s \subseteq L^t$.

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